

DOUBLE QUANTIZATION OF SEIBERG–WITTEN GEOMETRY AND W-ALGEBRAS

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ABSTRACT. We show that the double quantization of Seiberg–Witten spectral curve for Γ -quiver gauge theory defines the generating current of $W(\Gamma)$ -algebra in the free field realization. We also show that the partition function is given as a correlator of the corresponding $W(\Gamma)$ -algebra, which is equivalent to the AGT relation under the gauge/quiver (spectral) duality.

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1. INTRODUCTION AND SUMMARY

Recent progress on non-perturbative aspects of supersymmetric gauge theory exhibits interesting connections with various concepts of mathematical physics. In particular, Seiberg–Witten theory for four-dimensional $\mathcal{N} = 2$ gauge theory [1, 2] provides a geometrical interpretation for the Coulomb branch moduli space as a basis of the algebraic integrable system [3–6]. As a result, the Seiberg–Witten curve can be identified as a classical spectral

curve of the affine Toda chain for the pure SYM theory, the $SU(2)$ rational spin chain for $\mathcal{N} = 2$ SQCD with the fundamental hypermultiplets [7].

The relation between gauge theory and integrable system has various generalizations:

Dimensional hierarchy

Replacing the spacetime on which the gauge theory is defined with five and six dimensional manifolds compactified on S^1 and T^2 , it corresponds to trigonometric and elliptic deformations of rational integrable system [8, 9], as shown in Table 1. The Seiberg–Witten curve for 4d/5d/6d theory coincides with the spectral curve of rational/trigonometric/elliptic integrable system.

Quiver gauge theory

In the relation to integrable system, the gauge group rank is translated into the spin chain length. This means that change of the gauge symmetry doesn't lead to change of underlying symmetry on the integrable system side. For this purpose, we need quiver gauge theory labeled by a quiver graph Γ . The corresponding symmetry group for integrable system turns out to be G_Γ whose Dynkin diagram is given by Γ , and the Seiberg–Witten geometry is described by the fundamental characters of G_Γ group [9].

Ω -background and quantization

The four-dimensional spacetime for gauge theory allows an equivariant deformation with Ω -background [10, 11], and its specific limit $(\epsilon_1, \epsilon_2) \rightarrow (\hbar, 0)$, called the Nekrasov–Shatashvili (NS) limit, corresponds to quantization of integrable system [12, 13]. Then the Seiberg–Witten curve characterized by the algebraic relation is promoted to difference equation, and this quantum spectral curve is identified as Baxter's TQ-relation for G_Γ -spin chain, which is also equivalent to the corresponding Bethe ansatz equation [14] described by the polynomial equation called the q -character [15] (Table 2).

From this point of view, we still have another deformation parameter: We have used only one of equivariant parameters to quantize the integrable system. Thus it is natural to

Gauge theory geometry	Geometric realization	Integrable system
\mathbb{R}^4	IIA string on CY3	rational
$\mathbb{R}^4 \times S^1$	M theory on CY3 $\times S^1$	trigonometric
$\mathbb{R}^4 \times T^2$	F theory on CY3 $\times T^2$	elliptic

TABLE 1. Gauge theory geometry and IIA/M/F theory realization associated with rational/trigonometric/elliptic integrable systems. See, for example, [16].

Ω -background	SW geometry	Integrable system
$(\epsilon_1, \epsilon_2) = (0, 0)$	character	classical
$(\epsilon_1, \epsilon_2) = (\hbar, 0)$	q -character	quantum
$(\epsilon_1, \epsilon_2) \neq (0, 0)$ (w/ t -extention)	qq -character	doubly quantum (W-algebra)

TABLE 2. Ω -background and (double) quantization of Seiberg–Witten geometry described by G_Γ -character/ q -character/ qq -character for Γ -quiver gauge theory.

ask *what is the underlying algebraic structure for generic Ω -background parametrization*. It has been recently shown that the polynomial equation still holds for generic (ϵ_1, ϵ_2) just by replacing the q -character with another algebraic structure, called the qq -character [17, 18]. See also [19–26] for its realization in gauge theory and topological string theory.

We proceed with this program to construct W-algebras using quiver gauge theory [27, 28]. In our construction, in addition to consider generic equivariant parameters (ϵ_1, ϵ_2) , we incorporate the higher time variables, which induce deformation of UV prepotential by the local operators [29]. Our claim is as follows:

The qq -character together with double quantization of Seiberg–Witten spectral curve for the t -extended Γ -quiver gauge theory defines the generating current of $W(\Gamma)$ -algebra.

The doubly quantum spectral curve is an operator relation, in contrast to the classical and quantum curves, which give a relation between correlation functions. This double quantization scheme will be discussed in Sec. 3 in more detail.

Another claim about the partition function is:

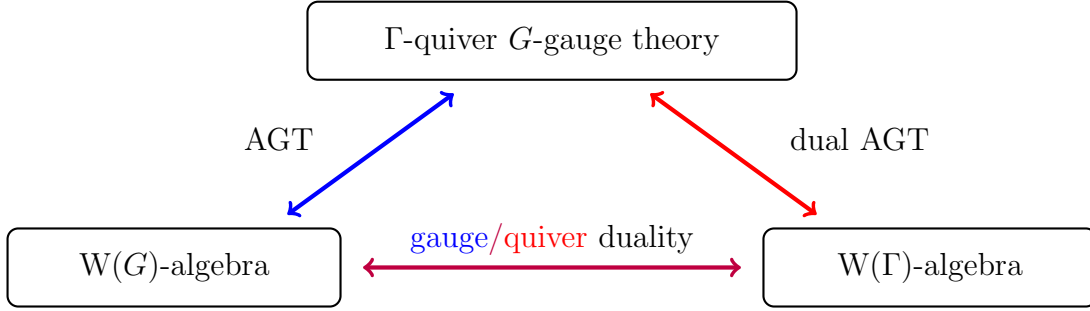


FIGURE 1. AGT and dual AGT

The t -extended Γ -quiver partition function is a state (Z -state) in the Hilbert space, and the non- t -extended partition function is given by a correlation function of $W(\Gamma)$ -algebra.

This statement is analogous, but slightly different from the AGT relation, which claims that the partition function of gauge theory enjoying gauge symmetry G is given by a correlation function of $W(G)$ -algebra [30, 31].¹ This reflects that our construction of $W(\Gamma)$ is equivalent to AGT for $W(G)$ by the gauge/quiver duality $G \leftrightarrow \Gamma$ [33–35], which is confirmed for A -type quiver (Fig. 1). In the classical limit, this duality becomes the Nahm–Fourier–Mukai duality between the phase space of Hitchin system on a cylinder and the phase space of periodic monopoles [9]. We remark a similar realization of W -algebra associated with the quiver structure [36–38].

Let us summarize our construction of quiver W -algebras [27, 28]. We start with the classical Seiberg–Witten curve given in the limit $(\epsilon_1, \epsilon_2) \rightarrow (0, 0)$. For Γ -quiver gauge theory, this spectral curve is given by the fundamental characters of G_Γ group, if the quiver diagram Γ coincides with a Dynkin diagram for finite simple Lie algebras. This construction is also applicable to generic quiver which doesn't correspond to any finite Lie algebra.² Tuning on the equivariant parameter as $(\epsilon_1, \epsilon_2) \rightarrow (\hbar, 0)$, we obtain the quantum Seiberg–Witten curve, which is identified as the TQ-relation of the integrable G_Γ -spin chain. We then consider the double quantization of Seiberg–Witten curve by taking into account the higher time variables with the fully generic equivariant parameters (ϵ_1, ϵ_2) . We obtain an operator

¹ Our construction naturally gives rise to the q -deformation of $W(\Gamma)$ -algebra from 5d quiver gauge theory, while the ordinary AGT relation claims a connection between 4d gauge theory and $W(G)$ -algebra without q -deformation. The counterpart of our construction is thus the q -AGT relation for 5d gauge theory and the q -deformed $W(G)$ -algebra [32].

²We assume in this article that the quiver is simply-laced.

relation, which is identified as the quantum Sugawara construction of the generating current of $W(\Gamma)$ -algebra.

Our construction exhibits several new features of W -algebras:

Edge (bifundamental) mass-deformation of W -algebra

In general, quiver gauge theory consists of the vectormultiplet assigned to a node of quiver i , and the hypermultiplet in bifundamental representation for an edge e with multiplicative mass parameter μ_e . (We will deal with (anti)fundamental hypermultiplets separately.) We will show that the bifundamental mass gives rise to new mass-deformation of W -algebra, which is reduced to Frenkel–Reshetikhin’s construction in the limit $\mu_e \rightarrow 1$ [39, 40].

Affine quiver W -algebra

Our construction of W -algebra is applicable to generic quiver gauge theory. Applying this construction to affine quiver theory, e.g., $\mathcal{N} = 2^*$ theory (\hat{A}_0 -quiver), we obtain new W -algebra associated to affine Lie algebra. The Seiberg–Witten curve (and its quantization) is described by the affine character, which is given as an infinite sum.

Quiver elliptic W -algebra

We can construct elliptically deformed W -algebras using six-dimensional quiver gauge theory on $\mathbb{R}^4 \times T^2$, which corresponds to the elliptic integrable system.

The rest of this article is organized as follows. We start in Sec. 2 to discuss the classical Seiberg–Witten geometry and its 1st quantization. The resultant quantum spectral curve obtained in the NS limit is a difference operator, which turns out to be Baxter’s TQ-relation, and the Q -operator plays a role of the wavefunction. In Sec. 3 we discuss 2nd quantization of the spectral curve. To obtain an operator relation, we introduce the higher time variable as a Heisenberg oscillator, corresponding to modification of the UV prepotential. We then show in Sec. 4 how the generating current of q -deformed Virasoro algebra arises in 5d gauge theory. The construction using the qq -character yields the free field realization of the current.

In Sec. 5 we show that the t -extended gauge theory partition function is given by a state, which we call the Z -state, in the Hilbert space through the operator/state correspondence. In particular, the Z -state is constructed with the screening charge of the corresponding conformal algebra. We also show that the (anti)fundamental matter contribution is realized by a vertex operator applied to the state. In Sec. 6 we show how to construct $W(\Gamma)$ -algebra associated with generic quiver Γ from the mass-deformed q -Cartan matrix characterizing the quiver gauge theory. We discuss the regularity of the qq -character which provides the generating current, and show it commutes with the screening charge of $W(\Gamma)$ -algebra. We demonstrate our construction with several examples reproducing known results in the limit. In Sec. 7 our construction is applied to affine quiver theory, involving the adjoint matter, corresponding to 4d $\mathcal{N} = 2^*$ (5d $\mathcal{N} = 1^*$) theory. This quiver doesn't correspond to finite Lie algebra, and thus provides a new W -algebra such that the generating current is given by an infinite sum of the fundamental current. In Sec. 8 we discuss elliptic generalization of quiver W -algebra using 6d gauge theory defined on $\mathbb{R}^4 \times T^2$. The whole construction discussed above is straightforwardly lifted, and thus we obtain elliptically deformed W -algebras associated with generic quiver.

2. SEIBERG–WITTEN SPECTRAL CURVE AND 1ST QUANTIZATION

The low-energy effective dynamics of 4d $\mathcal{N} = 2$ gauge theory is completely determined by the algebraic curve called the Seiberg–Witten curve [1, 2]. For $U(n)$ SYM theory it is characterized by the algebraic relation

$$y(x) + \frac{1}{y(x)} = T_n(x) \quad (2.1)$$

where $T_n(x)$ is a degree n monic polynomial in x^{-1} variable,³ and the vev of Coulomb moduli parameter is encoded into its coefficients.⁴ This algebraic relation can be obtained from the saddle point analysis of gauge theory partition function [9, 11, 41], and its derivation is quite analogous to the large N analysis of matrix models: the y -variable is realized as a one-point function of a generating function of gauge invariant observables (the Y -operator) [14]

$$y(x) = \left\langle Y(x) \right\rangle, \quad (2.3)$$

³While we usually use the convention such that $T_n(x)$ is a polynomial in x , we apply the opposite convention in this article for latter convenience.

⁴In this article we basically use the 5d notation:

$$q_1 = e^{\epsilon_1}, \quad q_2 = e^{\epsilon_2}, \quad q = q_1 q_2, \quad \{\nu_{i,n}\}_{i \in \Gamma_0, \alpha \in [1, \dots, n_i]} = \{e^{a_{i,n}}\}_{i \in \Gamma_0, \alpha \in [1, \dots, n_i]}. \quad (2.2)$$

These parameters correspond to the standard notation of q -deformed W -algebras: $(q_1, q_2) = (t^{-1}, q)$.

which plays a similar role to the resolvent function having a singularity in the complex plane. Incorporating the one-form differential defined on the curve, called the Seiberg–Witten differential,⁵

$$\lambda = \begin{cases} (4d) & x d \log y & (x, y) \in \mathbb{C} \times \mathbb{C}^\times \\ (5d) & \log x d \log y & (x, y) \in \mathbb{C}^\times \times \mathbb{C}^\times \end{cases}, \quad (2.4)$$

they define the Seiberg–Witten geometry describing the SUSY vacuum structure. We remark that the curve given by (2.1) coincides with the classical spectral curve of the affine n -site Toda chain, and also the inhomogeneous limit of $SU(2)$ rational/trigonometric/elliptic spin chain for 4d/5d/6d gauge theory.

Although the spectral curve is obtained in the classical limit $\epsilon_1, \epsilon_2 \rightarrow 0$ ($q_1, q_2 \rightarrow 1$), a similar structure can be found even after turning on the equivariant parameters. Taking into account one of the equivariant parameters $(\epsilon_1, \epsilon_2) \rightarrow (\hbar, 0)$, which is called the Nekrasov–Shatashvili limit, we obtain a difference equation instead of the algebraic relation for 5d theory [13, 14]⁶

$$y(x) + \frac{1}{y(q_1^{-1}x)} = T_n(x). \quad (2.5)$$

This is again obtained by the saddle point analysis with respect to $\epsilon_2 \rightarrow 0$, and the other finite parameter $\epsilon_1 = \hbar$ plays a role of the Planck constant. Indeed the difference equation (2.5) is interpreted as a quantization of the spectral curve (2.1) with respect to a canonical pair $(\log x, \log y)$,

$$[\log y, \log x] = \hbar, \quad (2.6)$$

which corresponds to the one-form (2.4) or the associated symplectic two-form

$$\Omega = d \log x \wedge d \log y. \quad (2.7)$$

This quantization implies the y -variable behaves as a shift operator $y = \exp(\hbar \partial_{\log x})$, and thus the classical curve (2.1) becomes a difference operator

$$[e^{\hbar \partial_{\log x}} + e^{-\hbar \partial_{\log x}} - T_n(x)] Q(x) = 0, \quad (2.8)$$

⁵The higher-dimensional lift is equivalent to imposing periodicity: The cylindrical coordinate for 5d theory on $\mathbb{R}^4 \times S^1$ is given by $\mathbb{C}^\times = \mathbb{C}/\mathbb{Z}$. The geometry for 6d gauge theory compactified on a torus $\mathbb{R}^4 \times T^2$ is parametrized by a coordinate $(x, y) \in E_{T^2} \times \mathbb{C}^\times$ where the elliptic curve E_{T^2} is doubly periodic, $E_{T^2} = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ with the torus modulus τ .

⁶This is also called the q -character due to its representation theoretical origin in quantum algebra [15].

which is equivalent to (2.5) under identification⁷

$$y(x) = \frac{Q(q_1x)}{Q(x)}. \quad (2.9)$$

In this way, quantization of spectral curve gives a differential/difference equation. In particular, in this case, the quantum curve (2.8) is equivalent to Baxter's TQ-relation for $SU(2)$ spin chain,⁸ which is a consequence of the connection between gauge theory and integrable system at the quantum level.

In addition, it has been recently revealed that a similar relation holds even away from the saddle point analysis, which is called the qq -character equation [17, 19, 22, 24, 25]

$$y(x) + \frac{1}{y(q^{-1}x)} = T_n(x). \quad (2.10)$$

We remark that, in the unrefined limit $q_1q_2 = q = 1$ ($\epsilon_1 + \epsilon_2 = 0$), it is not distinguishable from the classical relation (2.1), since the summation plays a role of the Planck constant in this case, $\epsilon_1 + \epsilon_2 = \hbar$. However, when we consider the higher-weight character, an extra factor has to be taken into account as discussed later. By the analogy with the matrix model and other related theories, this is quite natural for quantum curve. For example, in the case of the matrix model, the characteristic polynomial average plays a similar role to the wavefunction $Q(x)$ in (2.8), and actually satisfies the quantum curve equation at finite N under identification $\hbar = 1/N$. In general, quantization scheme is a way to get back to the finite coupling regime from the classical limit.

3. OPERATOR FORMALISM AND 2ND QUANTIZATION

As mentioned above, the y -variable plays a similar role to the resolvent function, and thus the classical spectral curve (2.1) and its quantizations (2.5) and (2.10) yield relations between (one-point) correlation functions in gauge theory. To emphasize it, let us rewrite the qq -character with the Y -operator

$$\langle Y(x) \rangle + \left\langle \frac{1}{Y(q^{-1}x)} \right\rangle = T_n(x). \quad (3.1)$$

Now the degree of $T_n(x)$ depends on the gauge group rank, which is not universal. However, the meaning of average $\langle \mathcal{O}(x) \rangle$ actually depends on the theory that we consider (gauge

⁷This transformation is analogous to that for the Riccati-type differential equation, widely used to derive quantum spectral curves.

⁸ To obtain a precise agreement, we have to take into account the Chern–Simons factor and the (anti)fundamental hypermultiplets. See [13, 14] for more arguments.

group, matter content, and so on). From this point of view, it is natural to define T -operator whose average with $U(n)$ SYM theory is given by

$$\langle T(x) \rangle = T_n(x). \quad (3.2)$$

Then we obtain a universal operator relation

$$Y(x) + \frac{1}{Y(q^{-1}x)} = T(x). \quad (3.3)$$

This is the 2nd quantization of Seiberg–Witten spectral curve in our sense, and this equation defines the T -operator in terms of the Y -operator.

Precisely speaking, in order to obtain a proper operator relation, we need to take into account the higher time variables, namely the potential term in gauge theory

$$Z_{\text{pot}}(t) = \exp \left(\sum_{n=1}^{\infty} t_n Y^{[n]} \right), \quad (3.4)$$

so that the t -extended partition function plays a role of generating function of the gauge theory observable $Y^{[n]}$, which is a trace of the n -th power adjoint scalar $\text{Tr } \Phi^n$, or its loop operator along a compactified circle for 5d theory. Inclusion of the potential term corresponds to deformation of the UV prepotential with the holomorphic operators [29]

$$\mathcal{F}_{\text{UV}} \longrightarrow \mathcal{F}_{\text{UV}} + \sum_{n=1}^{\infty} \frac{t_n}{n} \text{Tr } \Phi^n. \quad (3.5)$$

Then since the situation is completely parallel with the matrix model, let us explain the reason why we need the potential term (3.4) by the analogy. The resolvent operator of matrix model is a generating function of the gauge invariant observable, which is a single trace operator,

$$\text{Tr } \frac{1}{x - X} = \sum_{n=0}^{\infty} x^{-n-1} \text{Tr } X^n. \quad (3.6)$$

This observable is equivalent to derivative with respect to the coupling constant, playing a role of the time variable, in the potential term

$$\frac{\partial}{\partial t_n} e^{\text{Tr } V(X)} = \text{Tr } X^n e^{\text{Tr } V(X)} \quad (3.7)$$

where

$$V(x) = \sum_{n=1}^{\infty} t_n x^n. \quad (3.8)$$

In this sense the single trace observable is replaced with the derivative with t_n under the presence of the potential term.

4. FROM DOUBLE QUANTIZATION TO VIRASORO/W-ALGEBRA

In terms of the gauge theory observable, the \mathbf{Y} -operator (in particular, its average) is obtained [9, 14]

$$\langle \mathbf{Y}(x) \rangle = \exp \left(- \sum_{n=1}^{\infty} \frac{x^{-n}}{n} \mathbf{Y}^{[n]} \right). \quad (4.1)$$

Taking into account the higher time variables, the \mathbf{Y} -operator shows t -dependence, which is written in terms of (q_1, q_2) -modified free field operators [27]

$$\mathbf{Y}(x) = q_1^{\frac{1}{2}} : \exp \left(y_0 + \sum_{n \neq 0} y_n x^{-n} \right) : \quad (4.2)$$

where the free fields are defined

$$(n > 0) \quad y_{-n} = \frac{(1 - q_1^n)(1 - q_2^n)}{1 + q^n} t_n, \quad y_n = -\frac{1}{n} \frac{\partial}{\partial t_n}, \quad y_0 = -\frac{1}{2} t_0 \log q_2. \quad (4.3)$$

A constant and zero mode factors are added for convenience. According to the operator relation (3.3), it defines the T -operator in the free field realization, and it turns out to coincide with the generating current of q -deformed Virasoro algebra [39, 42]. Indeed the generator appearing in the mode expansion

$$T(x) = \sum_{n \in \mathbb{Z}} T_n x^{-n} \quad (4.4)$$

satisfies the algebraic relation

$$[T_n, T_m] = - \sum_{k=1}^{\infty} f_k (T_{n-k} T_{m+k} - T_{m-k} T_{n+k}) - \frac{(1 - q_1)(1 - q_2)}{1 - q} (q^n - q^{-n}) \delta_{n+m,0} \quad (4.5)$$

where

$$\sum_{k=0}^{\infty} f_k x^k = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1 - q_1^n)(1 - q_2^n)}{1 + q^n} x^n \right) =: f(x). \quad (4.6)$$

In practice, it is convenient to see the q -analog of OPE from the two-point function of the generating current, which is equivalent to the algebraic relation (4.5),

$$f\left(\frac{y}{x}\right) T(x) T(y) - f\left(\frac{x}{y}\right) T(y) T(x) = -\frac{(1 - q_1)(1 - q_2)}{1 - q} \left(\delta\left(\frac{qy}{x}\right) - \delta\left(\frac{qx}{y}\right) \right) \quad (4.7)$$

where the delta function is defined

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n. \quad (4.8)$$

This shows that the double quantization of Seiberg–Witten spectral curve provides the generating current of q -Virasoro algebra in the free field realization. We remark that such an

idea on the realization of generating current through the quantization of Seiberg–Witten curve was already discussed in the context of the (non q -deformed) AGT relation [30].

What we should remark is that the conformal algebra obtained here using the gauge theory does not depend on the gauge group rank: we obtain (q -deformed) Virasoro algebra for $U(n)$ gauge theory for arbitrary n . This situation is actually different from the AGT relation where the underlying conformal algebra is associated to the gauge group.

5. Z -STATE

Let us see how the gauge theory partition function is interpreted in our construction. In particular, the partition function extended with the time variables (3.4) explicitly depends on them. Since the time variable behaves as an operator, the t -extended partition function $Z(t)$ has to be identified as an operator. This interpretation directly leads to the notion of Z -state through the operator/state correspondence of CFT.

We show that the t -extended full partition function (including both one-loop and instanton contributions) for $U(n)$ SYM theory is a state given by [27]

$$|Z\rangle = \sum_{\mathcal{X} \in \mathfrak{M}^\top} \prod_{x \in \mathcal{X}}^{\succ} S(x) |1\rangle \quad (5.1)$$

where the product is a radial-ordered product, and \mathfrak{M}^\top is the torus action fixed point in the instanton moduli space, characterized by a set of partitions [11] with

$$\mathcal{X} = \{x_{\alpha,k}\}_{\alpha \in [1\dots n], k \in [1\dots \infty]}, \quad x_{\alpha,k} = \nu_\alpha q_1^{k-1} q_2^{\lambda_{\alpha,k}}. \quad (5.2)$$

The vacuum state is denoted by $|1\rangle$, which is a constant with respect to the time variables, $\partial_{t_n} |1\rangle = 0$, namely a primary state, and $S(x)$ is a screening current operator

$$S(x) = : \exp \left(s_0 \log x + \tilde{s}_0 + \frac{\kappa}{2} (\log_{q_2}^2 x - \log_{q_2} x) + \sum_{n \neq 0} s_n x^{-n} \right) :. \quad (5.3)$$

As well as the Y -operator, they are (q_1, q_2) -deformed free field operators

$$(n > 0) \quad s_{-n} = (1 - q_1^n) t_n, \quad s_n = -\frac{1 + q^{-n}}{n(1 - q_2^{-n})} \frac{\partial}{\partial t_n}, \quad s_0 = t_0 \quad (5.4)$$

obeying the commutation relation

$$[s_n, s_m] = -\frac{1}{n} \frac{1 - q_1^n}{1 - q_2^{-n}} (1 + q^{-n}) \delta_{n+m,0}. \quad (5.5)$$

The zero mode t_0 corresponds to the gauge coupling $t_0 = \log_{q_2} \mathbf{q}$, but we need some redefinition to obtain precise agreement with the gauge theory result. The additional zero mode

satisfies

$$[\tilde{s}_0, s_n] = -2\beta\delta_{n,0} \quad (5.6)$$

where $\beta = -\epsilon_1/\epsilon_2$, and the κ -term is added to reproduce the Chern–Simons term in gauge theory. See [27] for details.

The Z -state can be also written using the screening charge, which commutes with the generating current $T(x)$. First of all, the summation over the fixed points \mathfrak{M}^Γ in the partition function can be replaced with $\mathbb{Z}^{\mathcal{X}_0}$, which is a set of arbitrary integer sequences terminating by zeros. \mathcal{X}_0 is the ground configuration $\lambda_{\alpha,k} = 0$ for $k \in \mathbb{Z}_{>0}$, and $\dot{x}_{\alpha,k} = \nu_\alpha q_1^{k-1}$. If the partition $\lambda_{\alpha,k}$ is not a non-increasing sequence, there appears a zero factor in the expression

$$\prod_{x \in \mathcal{X}}^{\succ} S(x) |1\rangle = 0 \quad (5.7)$$

for $\mathcal{X} \in \mathbb{Z}^{\mathcal{X}_0}$ but $\mathcal{X} \notin \mathfrak{M}^\Gamma$. Therefore the Z -state is given by

$$|Z\rangle = \prod_{\dot{x} \in \mathcal{X}_0}^{\succ} \mathsf{S}(\dot{x}) |1\rangle \quad (5.8)$$

where we define the screening charge⁹

$$\mathsf{S}(\dot{x}) = \sum_{k \in \mathbb{Z}} S(q_2^k \dot{x}). \quad (5.10)$$

For q -deformed Virasoro/W-algebra, the screening charge is a discrete sum of the screening current. Precisely speaking, this sum is given by the Jackson integral to obtain a proper convergence property.

Let us mention an important property of the screening charge. The OPE between the Y -operator and the screening current is given by

$$\mathsf{Y}(x)S(x') = \frac{1-x'/x}{1-q_1x'/x} : \mathsf{Y}(x)S(x') : , \quad S(x')\mathsf{Y}(x) = q_1^{-1} \frac{1-x/x'}{1-q_1^{-1}x/x'} : \mathsf{Y}(x)S(x') : \quad (5.11)$$

and thus the commutation relation shows

$$[\mathsf{Y}(x), S(x')] = (1-q_1^{-1}) \delta\left(q_1 \frac{x'}{x}\right) : \mathsf{Y}(x)S(x') : \quad (5.12)$$

⁹This infinite sum can be written as the Jackson integral with respect to q_2 ,

$$\mathsf{S}(x) = \int dx_{q_2} S(x), \quad (5.9)$$

which is more convenient to discuss its convergence property. In the limit $q_2 \rightarrow 1$, the discrete sum is replaced with the usual integral.

where the delta function defined (4.8). Similarly we obtain

$$[\Upsilon(q^{-1}x)^{-1}, S(x')] = -(1 - q_1^{-1}) \delta\left(q \frac{x'}{x}\right) : \Upsilon(x) S(x') : \quad (5.13)$$

where we used the formula

$$q_1^{-1} : \Upsilon(x) \Upsilon(q^{-1}x) : = : S(q^{-1}x) S(q_1^{-1}x)^{-1} : \quad (5.14)$$

obtained by comparing the y and s fields, (4.3) and (5.4). We have almost the same OPEs for Υ and Υ^{-1} except for the q_2 -shift. This discrepancy is canceled in the screening charge because it is defined as a summation over the q_2 -shifted screening currents. Therefore the T -operator turns out to commute with the screening charge

$$[T(x), S(x')] = 0. \quad (5.15)$$

This assures the regularity of the T -operator, and the mode expansion (4.4). This commutativity also corresponds to the pole cancelation mechanism in the qq -character.

In order to obtain the non-extended plain partition function, we apply a dual vacuum state $\langle 1|$ because it plays a role of the projector into the $t = 0$ sector, $\langle 1| t_n = 0$. Thus the plain partition function is given by a correlator

$$Z(t = 0) = \langle 1| Z \rangle = \langle 1| \prod_{\check{x} \in \mathcal{X}_0}^{\succ} S(\check{x}) |1\rangle. \quad (5.16)$$

See also [36–38] for a similar construction using little string theory. This expression resembles the AGT relation [30], which is the coincidence between the gauge theory partition function and the Liouville/Toda CFT correlator. In the case of the AGT relation, we need to subtract the $U(1)$ factor to obtain agreement between the instanton partition function and the conformal block. In this case, on the other hand, we don't need to care about such an extra factor. We can directly see the agreement. In addition, as mentioned before, the underlying conformal algebra does not depend on the gauge group. We will come back to this issue later.

So far we have focused on the vector multiplet contribution. We can also incorporate the fundamental hypermultiplet in the operator formalism. Since the fixed point contribution

to the gauge theory observable is given by¹⁰

$$\mathbf{Y}^{[n]} = (1 - q_1^n) \sum_{x \in \mathcal{X}} x^n, \quad (5.18)$$

the potential term (3.4) yields the fundamental matter contribution by the shift of time variables

$$t_n \longrightarrow t_n + \frac{1}{n} \frac{1}{(1 - q_1^n)(1 - q_2^n)} \mu^{-n} \quad (5.19)$$

where μ is the multiplicative mass parameter. We introduce an operator which induces the time shift

$$\mathbf{V}(x) = : \exp \left(\sum_{n \neq 0} v_n x^{-n} \right) : \quad (5.20)$$

where

$$(n > 0) \quad v_{-n} = -\frac{1}{1 + q_1^n} t_n, \quad v_n = \frac{1}{n} \frac{1}{(1 - q_1^n)(1 - q_2^n)} \frac{\partial}{\partial t_n}. \quad (5.21)$$

We compute the OPE of \mathbf{V} and S operators

$$\mathbf{V}(x)S(x') = \left(\frac{x'}{x}; q_2 \right)_\infty^{-1} : \mathbf{V}(x)S(x') : , \quad S(x')\mathbf{V}(x) = \left(\frac{q_2 x}{x'}; q_2 \right)_\infty : \mathbf{V}(x)S(x') : . \quad (5.22)$$

They correspond to the fundamental and antifundamental hypermultiplet contributions, while the OPE of \mathbf{V} and \mathbf{V} does not yield dynamical contribution. Thus the extended partition function in the presence of (anti)fundamental matters is obtained by inserting the \mathbf{V} -operators

$$|Z\rangle = \left(\prod_{x \in \mathcal{X}_f} \mathbf{V}(x) \right) \left(\prod_{\check{x} \in \check{\mathcal{X}}_0} \mathbf{S}(\check{x}) \right) \left(\prod_{x \in \check{\mathcal{X}}_f} \mathbf{V}(x) \right) |1\rangle \quad (5.23)$$

where $\mathcal{X}_f = \{\mu_f\}_{f \in [1 \dots n^f]}$ and $\check{\mathcal{X}}_f = \{\tilde{\mu}_f\}_{f \in [1 \dots \check{n}^f]}$ are sets of fundamental and antifundamental mass parameters. This \mathbf{V} -operator puts a singularity on the spectral curve at $x = \mu_f$. Then the plain partition function is given by a correlator with additional vertex operators

$$Z(t=0) = \langle 1 | \left(\prod_{x \in \mathcal{X}_f} \mathbf{V}(x) \right) \left(\prod_{\check{x} \in \check{\mathcal{X}}_0} \mathbf{S}(\check{x}) \right) \left(\prod_{x \in \check{\mathcal{X}}_f} \mathbf{V}(x) \right) |1\rangle . \quad (5.24)$$

This correlator is given by an infinite discrete sum, which can be interpreted as a q -analog of Dotsenko–Fateev’s integral formula of the conformal block [43, 44].

¹⁰This expression can be interpreted as a q -deformation of n -th Casimir element

$$C_n = \frac{1}{n} \sum_{k=1}^{\infty} \left(\left(\lambda_k - k + \frac{1}{2} \right)^n - \left(-k + \frac{1}{2} \right)^n \right). \quad (5.17)$$

6. QUIVER W-ALGEBRA

The construction shown above is applicable to generic quiver gauge theory, and we correspondingly obtain the W-algebra associated to the quiver structure. Let us first fix the notations. Let Γ be a quiver with the set of nodes Γ_0 and the set of edges Γ_1 . A quiver Γ defines a $|\Gamma_0| \times |\Gamma_0|$ matrix

$$c_{ij}^{[n]} = (1 + q^{-n})\delta_{ij} - \sum_{e:j \rightarrow i} \mu_e^{-n} - \sum_{e:i \rightarrow j} \mu_e^n q^{-n} \quad (6.1)$$

where μ_e is the multiplicative bifundamental mass parameter associated to each edge $e \in \Gamma_1$. This is (q, μ_e) -deformation of Cartan matrix, reduced to the ordinary Cartan matrix in the limit $n \rightarrow 0$. In the analogy with matrix model, the quiver theory with gauge group $\times_i U(n_i)$ corresponds to the ADE multi-matrix model [45, 46], so that we have $|\Gamma_0|$ sets of *eigenvalues*

$$\mathcal{X}_i = \{x_{i,\alpha,k}\}_{\alpha \in [1 \dots n_i], k \in [1 \dots \infty]}, \quad \mathcal{X} = \sqcup_{i \in \Gamma_0} \mathcal{X}_i. \quad (6.2)$$

Let $i: \mathcal{X} \rightarrow \Gamma_0$ be the node label so that $i(x) = i$ for $x \in \mathcal{X}_i$.

Then the construction is quite parallel with the previous situation: The t -extended partition function is given by a state

$$|Z\rangle = \prod_{\hat{x} \in \mathcal{X}_0}^{\succ} S_{i(\hat{x})}(\hat{x}) |1\rangle. \quad (6.3)$$

The screening charge is defined as a summation over the screening current defined

$$S_i(x) = : \exp \left(s_{i,0} \log x + \tilde{s}_{i,0} + \frac{\kappa_i}{2} (\log_{q_2}^2 x - \log_{q_2} x) + \sum_{n \neq 0} s_{i,n} x^{-n} \right) :. \quad (6.4)$$

where the (q_1, q_2) modified free fields are written using the deformed Cartan matrix

$$(n > 0) \quad s_{i,-n} = (1 - q_1^n) t_{i,n}, \quad s_{i,n} = -\frac{1}{n(1 - q_2^{-n})} c_{ij}^{[n]} \frac{\partial}{\partial t_{j,n}}, \quad s_{i,0} = t_{i,0} \quad (6.5)$$

which obey the commutation relation

$$[s_{i,n}, s_{j,m}] = -\frac{1}{n} \frac{1 - q_1^n}{1 - q_2^{-n}} c_{ij}^{[n]} \delta_{n+m,0}. \quad (6.6)$$

Again the zero mode is related to the gauge coupling $t_{i,0} = \log_{q_2} \mathbf{q}_i$, under some redefinition. The additional zero mode similarly satisfies

$$[\tilde{s}_{i,0}, s_{j,n}] = -\beta \delta_{n,0} c_{ij}^{[0]}. \quad (6.7)$$

We can show that the Z -state (6.3) coincides with the t -extended gauge theory partition function [27].

Let us then discuss how the generating current is constructed using the Y -operator. For quiver gauge theory, the Y -operator is defined for each node $i \in \Gamma_0$, corresponding to the fundamental weight,

$$Y_i(x) = q_1^{\tilde{\rho}_i} : \exp \left(y_{i,0} + \sum_{n \neq 0} y_{i,n} x^{-n} \right) : \quad (6.8)$$

where $\tilde{\rho}_i = \sum_{j \in \Gamma_0} \tilde{c}_{ij}^{[0]}$ is the Weyl vector for non-affine quiver, and $\tilde{\rho}_i = 0$ for affine quiver. The modified free fields are given by¹¹

$$(n > 0) \quad y_{i,-n} = (1 - q_1^n)(1 - q_2^n) \tilde{c}_{ij}^{[-n]} t_{j,n}, \quad y_{i,n} = -\frac{1}{n} \frac{\partial}{\partial t_{i,n}}, \quad y_{i,0} = -\tilde{c}_{ij}^{[0]} t_{j,0} \log q_2 \quad (6.9)$$

with the commutation relation

$$\begin{aligned} [y_{i,n}, y_{j,m}] &= -\frac{1}{n} (1 - q_1^n)(1 - q_2^n) \tilde{c}_{ji}^{[-n]} \delta_{n+m,0} \\ &= -\frac{1}{n} (1 - q_1^{-n})(1 - q_2^{-n}) \tilde{c}_{ij}^{[n]} \delta_{n+m,0}. \end{aligned} \quad (6.10)$$

In this case, we can define the T -operator which starts with each operator $Y_i(x)$ for $i \in \Gamma_0$. It is generated by the local reflection, which is the mass deformed iWeyl reflection [17]¹²

$$T_i(x) = Y_i(x) + : Y_i(q^{-1}x)^{-1} \prod_{e:j \rightarrow i} Y_j(\mu_e q^{-1}x) \prod_{e:i \rightarrow j} Y_j(\mu_e^{-1}x) : + \cdots \quad (6.11)$$

where we assume that there is no self-connecting edge in the quiver. See Sec. 7 for \hat{A}_0 quiver. This reflection formula is obtained by comparing configurations at the fixed points which differ by one instanton factor. This means that we shall see the behavior under the partition shift $\lambda_{i,\alpha,k} \rightarrow \lambda_{i,\alpha,k} + 1$, which is equivalent to $x \rightarrow q_2 x$ for $x \in \mathcal{X}_i$. Since the Z -state (6.3) is constructed by the screening charges, we consider the shift of screening current,

$$: \frac{S_i(x)}{S_i(q_2 x)} : = q_1^{-1} A_i(x) \quad (6.12)$$

where the A -operator is defined

$$A_i(x) = q_1 : \exp \left(a_{i,0} + \sum_{n \neq 0} a_{i,n} x^{-n} \right) : \quad (6.13)$$

with the free field operators

$$a_{i,n} = (1 - q_2^{-n}) s_{i,n}, \quad a_{i,0} = -t_{i,0} \log q_2. \quad (6.14)$$

¹¹For the quiver whose Cartan matrix is not invertible $\det c^{[0]} = 0$, we cannot incorporate the coupling constant into the zero mode $t_{i,0} = \log_{q_2} q_i$. One needs to put the q_i factor for every iWeyl reflection by hand.

¹²In the classical limit $q_1, q_2, \mu_e \rightarrow 1$, this reflection is reduced to the ordinary Weyl reflection, and thus the classical Seiberg–Witten geometry is expressed using the character of group G_Γ , associated to Dynkin-quiver diagram Γ [9].

These \mathbf{A} and \mathbf{Y} operators correspond to “root” and “fundamental weight” respectively [40]. Thus the oscillators are related as follows:

$$a_{i,n} = c_{ij}^{[n]} y_{j,n}, \quad (6.15)$$

or explicitly,

$$\mathbf{A}_i(x) = : \frac{\mathbf{Y}_i(x) \mathbf{Y}_i(qx)}{\prod_{e:j \rightarrow i} \mathbf{Y}_j(\mu_e x) \prod_{e:i \rightarrow j} \mathbf{Y}_j(\mu_e^{-1} qx)} :. \quad (6.16)$$

This leads to the pole cancelation in the iWeyl reflection

$$\text{Res}_{x' \rightarrow q^{-1}x} \left[\mathbf{Y}_i(x) S_i(q_2 x') \right] + \text{Res}_{x' \rightarrow q^{-1}x} \left[: \mathbf{Y}_i(x) \mathbf{A}_i(q^{-1}x)^{-1} : S_i(x') \right] = 0. \quad (6.17)$$

Similarly we can show that the T -operator commutes with the screening charge defined as a summation over the screening operator (5.10),

$$\left[T_i(x), S_j(x') \right] = 0. \quad (6.18)$$

This shows that the T -operator yields the generating current of $W(\Gamma)$ -algebra associated to Γ -quiver in the free field realization.

We remark that when there appears a product of the \mathbf{Y} -operators from the same node $i \in \Gamma_0$ in the reflection (6.11), one needs an extra structure

$$: \mathbf{Y}_i(x) \mathbf{Y}_i(x') : + \mathcal{S} \left(\frac{x'}{x} \right) : \frac{\mathbf{Y}_i(x) \mathbf{Y}_i(x')}{\mathbf{A}_i(q^{-1}x)} : + \mathcal{S} \left(\frac{x}{x'} \right) : \frac{\mathbf{Y}_i(x) \mathbf{Y}_i(x')}{\mathbf{A}_i(q^{-1}x')} : + : \frac{\mathbf{Y}_i(x) \mathbf{Y}_i(x')}{\mathbf{A}_i(q^{-1}x) \mathbf{A}_i(q^{-1}x')} : \quad (6.19)$$

where the factor corresponding to the OPE between \mathbf{Y} and \mathbf{A} operators defined

$$\mathcal{S}(x) = \frac{(1 - q_1 x)(1 - q_2 x)}{(1 - qx)(1 - x)}. \quad (6.20)$$

Furthermore, in the collision limit $x' \rightarrow x$, it involves a derivative term

$$: \mathbf{Y}_i(x)^2 : + : \left(\mathfrak{c}(q_1, q_2) - \frac{(1 - q_1)(1 - q_2)}{1 - q} \partial_{\log x} \log \mathbf{A}_i(q^{-1}x) \right) \frac{\mathbf{Y}_i(x)^2}{\mathbf{A}_i(q^{-1}x)} : + : \frac{\mathbf{Y}_i(x)^2}{\mathbf{A}_i(q^{-1}x)^2} : \quad (6.21)$$

where the constant defined

$$\mathfrak{c}(q_1, q_2) = \lim_{x \rightarrow 1} (\mathcal{S}(x) + \mathcal{S}(x^{-1})) = \frac{1 - 6q + q^2 + (q_1 + q_2)(1 + q)}{(1 - q)^2}. \quad (6.22)$$

In the NS limit $q_2 \rightarrow 1$, the derivative term vanishes, and the factor becomes $\mathfrak{c}(q_1, 1) = 2$ with $\mathcal{S}(x) \rightarrow 1$. Thus the reflection structure becomes simple.

We notice that the free fields (6.9) explicitly depend on the bifundamental mass parameter, which may affect the algebraic relation of W-algebra. Since our construction coincides with

the definition of W-algebra given by [40] in the mass less limit $\mu_e \rightarrow 1$ for $e \in \Gamma_1$, the algebra obtained from quiver gauge theory yields a new mass deformation of W-algebra. Furthermore, the bifundamental mass can be absorbed by redefinition of the Coulomb moduli parameter when there is no loop edge [9, 14].

6.1. A_1 quiver. We show more explicit examples. The simplest example is the A_1 quiver. The T -operator is given by

$$T_1(x) = \Upsilon_1(x) + \frac{1}{\Upsilon_1(q^{-1}x)}, \quad (6.23)$$

which is a double quantum deformation of the fundamental representation character for $G_\Gamma = SU(2)$

$$\chi_{\mathbf{2}}^{SU(2)}(y) = y + y^{-1}. \quad (6.24)$$

In addition, we compute the degree n current, corresponding to $(n/2)$ -spin representation of $SU(2)$, with the weight (w_1, \dots, w_n)

$$\begin{aligned} T_1^{[\mathbf{w}]}(x) &= : \Upsilon_1(w_1 x) \Upsilon_1(w_2 x) \cdots \Upsilon_1(w_n x) : + \cdots \\ &= \sum_{I \cup J = \{1 \dots n\}} \prod_{i \in I, j \in J} s\left(\frac{w_i}{w_j}\right) : \prod_{i \in I} \Upsilon_1(w_i x) \prod_{j \in J} \Upsilon_1(w_j q^{-1} x)^{-1} : . \end{aligned} \quad (6.25)$$

The degree two current is especially used to define the algebraic relation for the q -Virasoro algebra (4.7).

6.2. A_2 quiver. The next example is the A_2 quiver, consisting of two nodes and an edge connecting them. We have two T -operators

$$T_1(x) = \Upsilon_1(x) + : \frac{\Upsilon_2(\mu^{-1}x)}{\Upsilon_1(q^{-1}x)} : + \frac{1}{\Upsilon_2(\mu^{-1}q^{-1}x)}, \quad (6.26)$$

$$T_2(x) = \Upsilon_2(x) + : \frac{\Upsilon_1(\mu q^{-1}x)}{\Upsilon_2(q^{-1}x)} : + \frac{1}{\Upsilon_1(\mu q^{-2}x)}, \quad (6.27)$$

where $\mu = \mu_{1 \rightarrow 2} = \mu_{2 \rightarrow 1}^{-1} q$. They correspond to the fundamental and antifundamental representation characters of $G_\Gamma = SU(3)$

$$\chi_{\mathbf{3}}^{SU(3)}(y_1, y_2) = y_1 + y_1^{-1} y_2 + y_2^{-1}, \quad \chi_{\bar{\mathbf{3}}}^{SU(3)}(y_1, y_2) = y_2 + y_1 y_2^{-1} + y_1^{-1}. \quad (6.28)$$

The OPEs for these T -operators are given by

$$\begin{aligned} f_{11} \left(\frac{y}{x} \right) T_1(x) T_1(y) - f_{11} \left(\frac{x}{y} \right) T_1(y) T_1(x) \\ = - \frac{(1-q_1)(1-q_2)}{1-q} \left(\delta \left(q \frac{y}{x} \right) T_2(\mu^{-1}x) - \delta \left(q^{-1} \frac{y}{x} \right) T_2(\mu^{-1}qx) \right) \end{aligned} \quad (6.29)$$

$$\begin{aligned} f_{12} \left(\frac{y}{x} \right) T_1(x) T_2(y) - f_{21} \left(\frac{x}{y} \right) T_2(y) T_1(x) \\ = - \frac{(1-q_1)(1-q_2)}{1-q} \left(\delta \left(\mu q \frac{y}{x} \right) - \delta \left(\mu q^{-2} \frac{y}{x} \right) \right) \end{aligned} \quad (6.30)$$

$$\begin{aligned} f_{22} \left(\frac{y}{x} \right) T_2(x) T_2(y) - f_{22} \left(\frac{x}{y} \right) T_2(y) T_2(x) \\ = - \frac{(1-q_1)(1-q_2)}{1-q} \left(\delta \left(q \frac{y}{x} \right) T_1(\mu q^{-1}x) - \delta \left(q^{-1} \frac{y}{x} \right) T_1(\mu x) \right) \end{aligned} \quad (6.31)$$

where

$$f_{ij}(x) = \exp \left(\sum_{n=1}^{\infty} \frac{(1-q_1^n)(1-q_2^n)}{n} \tilde{c}_{ji}^{[-n]} x^n \right). \quad (6.32)$$

Now the symmetry under exchange $Y_1 \leftrightarrow Y_2$ is obvious. We can read off the algebraic relation for the (q, μ) -deformed $W(A_2)$ algebra from these OPEs, which explicitly depends on the mass parameter μ . We remark that the diagonal element $f_{ii}(x)$ does not depend on the mass parameter μ_e in general. It appears only in the off-diagonal element $f_{ij}(x)$ for $i \neq j$.

6.3. A_r quiver. Let us demonstrate our construction with the linear quiver theory $\Gamma = A_r$. The T -operators are computed using the iWeyl reflection

$$T_i(\mu_i^{-1}x) = \sum_{1 \leq j_1 \leq \dots \leq j_i \leq r+1} : \prod_{k=1}^i \Lambda_{j_k}(q^{-i+k}x) : \quad (6.33)$$

where

$$\Lambda_i(x) = Y_i(\mu_i^{-1}x) Y_{i-1}(\mu_{i-1}^{-1}q^{-1}x)^{-1} \quad (6.34)$$

with $Y_0(x) = Y_{r+1}(x) = 1$, and the mass product defined

$$\mu_i := \mu_{1 \rightarrow 2} \mu_{2 \rightarrow 3} \cdots \mu_{i-1 \rightarrow i}. \quad (6.35)$$

We remark that these T -operators are given by (q, μ) -deformation of fundamental representation characters for $G_\Gamma = SU(r+1)$. This construction coincides with the quantum Miura transformation for the q -deformed $W(A_r)$ -algebra in the massless limit $\mu_e \rightarrow 1$ [39, 47]

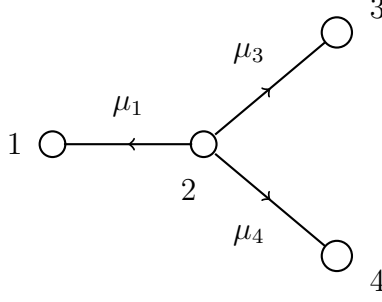


FIGURE 2. D_4 quiver decorated with bifundamental mass $\mu_i = \mu_{2 \rightarrow i}$.

6.4. D_r quiver. We consider D -type quiver gauge theory. The simplest case is $\Gamma = D_4$, shown in Fig. 2. This quiver shows a symmetry under exchange $1 \leftrightarrow 3 \leftrightarrow 4$, which is known as the $SO(8)$ triality. Putting the bifundamental mass parameters $\mu_i := \mu_{2 \rightarrow i}$ for $i = 1, 3, 4$, we compute the T -operator

$$\begin{aligned}
 T_1(x) = & \Upsilon_1(x) + : \frac{\Upsilon_2(\mu_1 q^{-1}x)}{\Upsilon_1(q^{-1}x)} : + : \frac{\Upsilon_3(\mu_1 \mu_3^{-1} q^{-1}x) \Upsilon_4(\mu_1 \mu_4^{-1} q^{-1}x)}{\Upsilon_2(\mu_1 q^{-2}x)} : + : \frac{\Upsilon_4(\mu_1 \mu_4^{-1} q^{-1}x)}{\Upsilon_3(\mu_1 \mu_3^{-1} q^{-2}x)} : \\
 & + : \frac{\Upsilon_3(\mu_1 \mu_3^{-1} q^{-1}x)}{\Upsilon_4(\mu_1 \mu_4^{-1} q^{-2}x)} : + : \frac{\Upsilon_2(\mu_1 q^{-2}x)}{\Upsilon_3(\mu_1 \mu_3^{-1} q^{-2}x) \Upsilon_4(\mu_1 \mu_4^{-1} q^{-2}x)} : + : \frac{\Upsilon_1(q^{-2}x)}{\Upsilon_2(\mu_1 q^{-3}x)} : + \frac{1}{\Upsilon_1(q^{-3}x)}.
 \end{aligned} \tag{6.36}$$

The operators $T_3(x)$ and $T_4(x)$ are similarly obtained by permutation. These three T -operators correspond to three **8**-representations of $SO(8)$. The remaining $T_2(x)$ operator, corresponding to **28**-representation, involves collision and derivative terms, (6.19) and (6.21),

$$\begin{aligned}
 T_2(x) = & T_2^+(x) + T_2^-(x) + \mathcal{S}(q) \left(: \frac{\Upsilon_1(\mu_1^{-1}x)}{\Upsilon_1(\mu_1^{-1} q^{-1}x)} : + : \frac{\Upsilon_3(\mu_3^{-1}x)}{\Upsilon_3(\mu_3^{-1} q^{-1}x)} : + : \frac{\Upsilon_4(\mu_4^{-1}x)}{\Upsilon_4(\mu_4^{-1} q^{-1}x)} : \right) \\
 & + \left(\mathfrak{c}(q_1, q_2) - \frac{(1-q_1)(1-q_2)}{1-q} \partial_{\log x} \log \left(\frac{\Upsilon_2(q^{-1}x) \Upsilon_2(q^{-2}x)}{\Upsilon_1(\mu_1^{-1} q^{-1}x) \Upsilon_3(\mu_3^{-1} q^{-1}x) \Upsilon_4(\mu_4^{-1} q^{-1}x)} \right) \right) : \frac{\Upsilon_2(q^{-1}x)}{\Upsilon_2(q^{-2}x)} :
 \end{aligned} \tag{6.37}$$

where

$$\begin{aligned}
T_2^+(x) = & Y_2(x) + : \frac{Y_1(\mu_1^{-1}x)Y_3(\mu_3^{-1}x)Y_4(\mu_4^{-1}x)}{Y_2(q^{-1}x)} : + : \frac{Y_3(\mu_3^{-1}x)Y_4(\mu_4^{-1}x)}{Y_1(\mu_1^{-1}q^{-1}x)} : + : \frac{Y_1(\mu_1^{-1}x)Y_4(\mu_4^{-1}x)}{Y_3(\mu_3^{-1}q^{-1}x)} : \\
& + : \frac{Y_1(\mu_1^{-1}x)Y_3(\mu_3^{-1}x)}{Y_4(\mu_4^{-1}q^{-1}x)} : + : \frac{Y_1(\mu_1^{-1}x)Y_2(q^{-1}x)}{Y_3(\mu_3^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)} : + : \frac{Y_3(\mu_3^{-1}x)Y_2(q^{-1}x)}{Y_1(\mu_1^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)} : \\
& + : \frac{Y_4(\mu_4^{-1}x)Y_2(q^{-1}x)}{Y_1(\mu_1^{-1}q^{-1}x)Y_3(\mu_3^{-1}q^{-1}x)} : + : \frac{Y_2(q^{-1}x)^2}{Y_1(\mu_1^{-1}q^{-1}x)Y_3(\mu_3^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)} : \\
& + : \frac{Y_1(\mu_1^{-1}x)Y_1(\mu_1^{-1}q^{-1}x)}{Y_2(q^{-2}x)} : + : \frac{Y_3(\mu_3^{-1}x)Y_3(\mu_3^{-1}q^{-1}x)}{Y_2(q^{-2}x)} : + : \frac{Y_4(\mu_4^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)}{Y_2(q^{-2}x)} :, \\
\end{aligned} \tag{6.38}$$

$$\begin{aligned}
T_2^-(x) = & \frac{Y_1(\mu_1^{-1}q^{-1}x)Y_3(\mu_3^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)}{Y_2(q^{-2}x)^2} : + : \frac{Y_2(q^{-1}x)}{Y_1(\mu_1^{-1}q^{-1}x)Y_1(\mu_1^{-1}q^{-2}x)} : \\
& + : \frac{Y_2(q^{-1}x)}{Y_3(\mu_3^{-1}q^{-1}x)Y_3(\mu_3^{-1}q^{-2}x)} : + : \frac{Y_2(q^{-1}x)}{Y_4(\mu_4^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-2}x)} : + : \frac{Y_3(\mu_3^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)}{Y_1(\mu_1^{-1}q^{-2}x)Y_2(q^{-2}x)} : \\
& + : \frac{Y_1(\mu_1^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)}{Y_3(\mu_3^{-1}q^{-2}x)Y_2(q^{-2}x)} : + : \frac{Y_1(\mu_1^{-1}q^{-1}x)Y_3(\mu_3^{-1}q^{-1}x)}{Y_4(\mu_4^{-1}q^{-2}x)Y_2(q^{-2}x)} : + : \frac{Y_1(\mu_1^{-1}q^{-1}x)}{Y_3(\mu_3^{-1}q^{-2}x)Y_4(\mu_4^{-1}q^{-2}x)} : \\
& + : \frac{Y_3(\mu_1^{-1}q^{-1}x)}{Y_1(\mu_1^{-1}q^{-2}x)Y_4(\mu_4^{-1}q^{-2}x)} : + : \frac{Y_4(\mu_4^{-1}q^{-1}x)}{Y_1(\mu_1^{-1}q^{-2}x)Y_3(\mu_3^{-1}q^{-2}x)} : \\
& + : \frac{Y_2(q^{-2}x)}{Y_1(\mu_1^{-1}q^{-2}x)Y_3(\mu_3^{-1}q^{-2}x)Y_4(\mu_4^{-1}q^{-2}x)} : + Y_2(q^{-3}x)^{-1} . \\
\end{aligned} \tag{6.39}$$

In the classical limit, this T_2 -operator is reduced to the character of 28 dimensional representation of $SO(8)$.

7. AFFINE QUIVER W-ALGEBRA

Applying our construction to affine quiver gauge theory, we can define a new W-algebra, associated to affine Lie algebra. Let us consider the simplest affine quiver $\Gamma = \hat{A}_0$, corresponding to 4d $\mathcal{N} = 2^*$ (5d $\mathcal{N} = 1^*$) theory, whose deformation of Cartan matrix is given by

$$c^{[n]} = 1 + q^{-n} - \mu^{-n} - \mu^n q^{-n} \xrightarrow{n \rightarrow 0} 0. \tag{7.1}$$

This Cartan matrix can be nontrivial only in the presence of the multiplicative adjoint mass parameter $\mu (\neq 1)$. This means that the mass deformation of the Cartan matrix is essential for the affine quiver, and this deformation goes beyond Frenkel–Reshetikhin’s construction

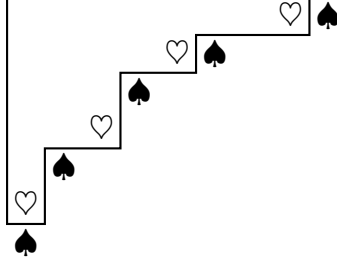


FIGURE 3. The outer and inner boundaries of the partition, $\partial_+\lambda$ and $\partial_-\lambda$, denote the to-be-added and to-be-removed boxes, ♠ and ♡, respectively. The number of ♠ is always bigger than ♡ by one.

of W-algebra [39, 40]. The free field operators for \mathbf{Y} and \mathbf{S} fields obey the commutation relation

$$\left[y_{1,n}, y_{1,m} \right] = -\frac{1}{n} \frac{(1 - q_1^n)(1 - q_2^n)}{(1 - \mu^n)(1 - \mu^{-n}q^n)} \delta_{n+m,0}, \quad (7.2)$$

and

$$\left[s_{1,n}, s_{1,m} \right] = -\frac{1}{n} \frac{1 - q_1^n}{1 - q_2^{-n}} (1 - \mu^{-n})(1 - \mu^n q^{-n}) \delta_{n+m,0}. \quad (7.3)$$

The s -oscillator becomes trivial in the limit $\mu \rightarrow 1$ (and also $\mu \rightarrow q$), which means that the Nekrasov factor becomes trivial in $\mathcal{N} = 4$ theory, while the y -oscillator becomes singular in this limit.

The T -operator in this case is given by the iWeyl reflection as well, but it does not terminate within a finite sum: the affine character needs an infinite sum parametrized by the partition

$$\begin{aligned} T_1(x) &= \mathbf{Y}_1(x) + \mathbf{q} \mathcal{S}(\mu^{-1}) : \mathbf{Y}_1(q^{-1}x)^{-1} \mathbf{Y}_1(\mu^{-1}x) \mathbf{Y}_1(\mu q^{-1}x) : + \cdots \\ &= \sum_{\lambda} \mathbf{q}^{|\lambda|} Z_{\lambda}^{\hat{A}_0}(\tilde{q}_1, \tilde{q}_2, \tilde{\mu}) : \prod_{s \in \partial_+\lambda} \mathbf{Y}_1(qx/\tilde{x}(s)) \prod_{s \in \partial_-\lambda} \mathbf{Y}_1(x/\tilde{x}(s))^{-1} : \end{aligned} \quad (7.4)$$

where the coupling constant \mathbf{q} is now explicitly appearing because the Cartan matrix $c = (0)$ is not invertible for affine quiver $\Gamma = \hat{A}_0$. $\partial_+\lambda$ and $\partial_-\lambda$ are the outer and inner boundary of the partition λ , as shown in Fig. 3, and we define

$$\tilde{x}(s) = (\mu^{-1}q)^{s_1-1} \mu^{s_2-1} q, \quad (7.5)$$

which reads the dual variables

$$\tilde{q}_1 = \mu^{-1}q, \quad \tilde{q}_2 = \mu, \quad \tilde{\mu} = q_2, \quad \tilde{\nu} = q. \quad (7.6)$$

We remark that $\tilde{q} := \tilde{q}_1 \tilde{q}_2 = q$. The Nekrasov instanton function $Z_{\lambda}^{\hat{A}_0}(\tilde{q}_1, \tilde{q}_2, \tilde{\mu})$ is again for \hat{A}_0 quiver, but $U(1)$ gauge theory

$$Z_{\lambda}^{\hat{A}_0}(q_1, q_2, \mu) = \prod_{(s_1, s_2) \in \lambda} \frac{(1 - \mu q_1^{l(s)+1} q_2^{-a(s)})(1 - \mu^{-1} q_1^{l(s)} q_2^{-a(s)-1})}{(1 - q_1^{l(s)+1} q_2^{-a(s)})(1 - q_1^{l(s)} q_2^{-a(s)-1})} \quad (7.7)$$

where the arm and leg lengths are defined: $a(s) = \lambda_{s_1} - s_2$, $l(s) = \lambda_{s_2}^t - s_1$. The higher degree current is given as a summation over multiple partition.

8. QUIVER ELLIPTIC W-ALGEBRA

We have constructed the q -deformed W-algebra $W_{q_1, q_2}(\Gamma)$ associated with Γ -quiver gauge theory in 5d, which corresponds to trigonometric (relativistic) integrable system. According to such a connection with integrable systems, it is natural to consider its elliptic deformation using 6d gauge theory compactified on the elliptic curve $\mathbb{R}^4 \times T^2$ [28]. See also [20, 23, 24, 48–53] for recent results along this direction.

In this case the spectral parameter is periodic

$$x \simeq px \quad (8.1)$$

where τ is the modulus of T^2 that defines $p = e^{2\pi i \tau}$. One needs to impose this periodicity on the index functor. Recalling that the 5d gauge theory partition function on $\mathbb{R}^4 \times S^1$ is given by the index along S^1 defined

$$\mathbb{I} \left[\sum_{\alpha} x_{\alpha} \right] = \prod_{\alpha} (1 - x_{\alpha}^{-1}) , \quad (8.2)$$

the elliptic index (equivariant elliptic genus on T^2) yields the 6d partition function for $\mathbb{R}^4 \times T^2$

$$\mathbb{I}_p \left[\sum_{\alpha} x_{\alpha} \right] = \prod_{\alpha} \theta(x_{\alpha}^{-1}; p) \quad (8.3)$$

where the Jacobi theta function is given by¹³

$$\begin{aligned}\theta(x; p) &= (x; p)_\infty (px^{-1}; p)_\infty \\ &= \exp \left(- \sum_{n \neq 0} \frac{1}{n} \frac{x^n}{1 - p^n} \right).\end{aligned}\tag{8.7}$$

We assume $|p| < 1$ and this elliptic system is reduced to 5d theory in the limit $p \rightarrow 0$.

Since the elliptic index (8.3) involves both positive and negative powers due to the expansion (8.7), we need to introduce “positive” and “negative” oscillators [28, 54, 55]

$$\left[y_{i,n}^{(\pm)}, y_{j,m}^{(\pm)} \right] = \mp \frac{1}{n} \frac{(1 - q_1^{\pm n})(1 - q_2^{\pm n})}{1 - p^{\pm n}} \tilde{c}_{ji}^{[\mp n]} \delta_{n+m,0},\tag{8.8}$$

$$\left[s_{i,n}^{(\pm)}, s_{j,m}^{(\pm)} \right] = \mp \frac{1}{n} \frac{1 - q_1^{\pm n}}{(1 - p^{\pm n})(1 - q_2^{\mp n})} c_{ij}^{[\pm n]} \delta_{n+m,0},\tag{8.9}$$

and thus

$$\mathbf{Y}_i(x) = q_1^{\tilde{\rho}_i} : \exp \left(y_{i,0} + \sum_{n \neq 0} y_{i,n}^{(+)} x^{-n} + \sum_{n \neq 0} y_{i,n}^{(-)} x^{+n} \right) :, \tag{8.10}$$

$$S_i(x) = : \exp \left(s_{i,0} \log x + \tilde{s}_{i,0} + \sum_{n \neq 0} s_{i,n}^{(+)} x^{-n} + \sum_{n \neq 0} s_{i,n}^{(-)} x^{+n} \right) :. \tag{8.11}$$

Then the remaining construction is completely parallel with 5d theory: The T -operator is generated by the iWeyl reflection with the \mathbf{Y} -operator, and the Z -state is constructed using the screening charge, built with the elliptic oscillators (8.8) and (8.9). In this procedure, the whole rational functions are replaced with the elliptic function. For example, the scalar factor (6.20) due to the \mathbf{Y} and \mathbf{A} OPE becomes

$$\mathcal{S}(x) = \frac{\theta(q_1 x; p) \theta(q_2 x; p)}{\theta(qx; p) \theta(x; p)}, \tag{8.12}$$

¹³In this article we apply the Dolbeault convention, which obeys

$$\mathbb{I} [x^{-1}] = -x^{-1} \mathbb{I} [x], \quad \mathbb{I}_p [x^{-1}] = -x^{-1} \mathbb{I}_p [x]. \tag{8.4}$$

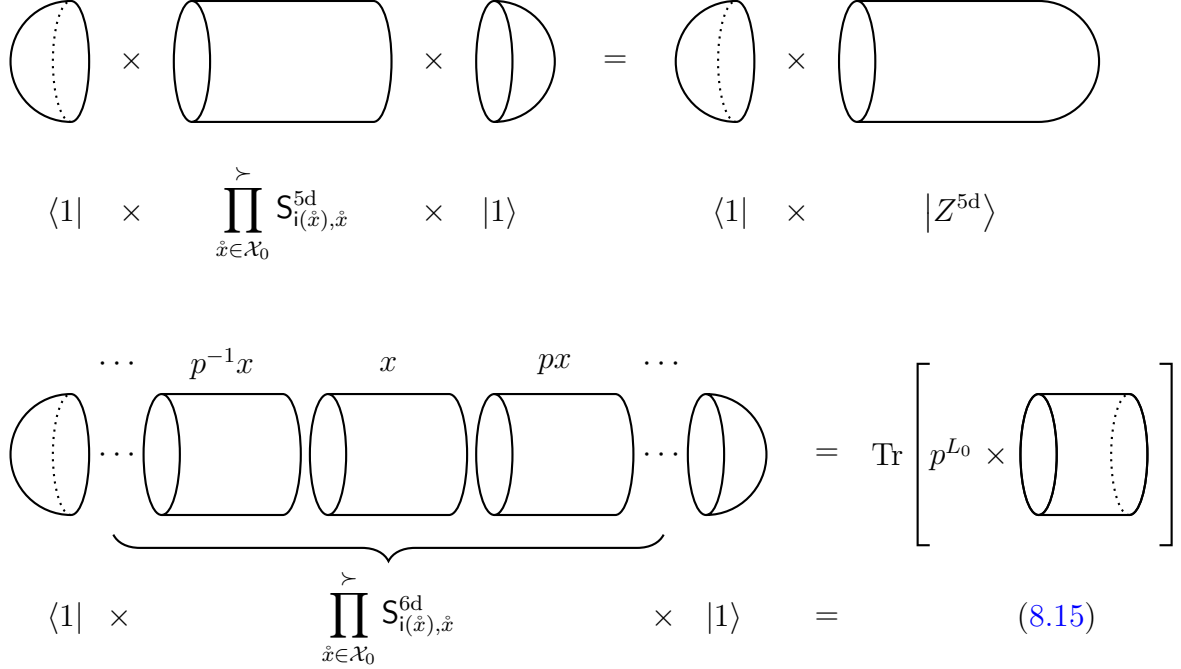
This convention can be converted into the Dirac index,

$$\mathbb{I} \left[\sum_{\alpha} x_{\alpha} \right] = \prod_{\alpha} \left(x_{\alpha}^{1/2} - x_{\alpha}^{-1/2} \right), \quad \mathbb{I}_p \left[\sum_{\alpha} x_{\alpha} \right] = \prod_{\alpha} \theta_1(x_{\alpha}; p) \tag{8.5}$$

with

$$\mathbb{I} [x^{-1}] = -\mathbb{I} [x], \quad \mathbb{I}_p [x^{-1}] = -\mathbb{I}_p [x]. \tag{8.6}$$

Under the conformal condition (8.14), these two conventions are consistent.



$$\begin{aligned}
& \langle 1| \times \prod_{\hat{x} \in \mathcal{X}_0}^{\succ} S_{i(\hat{x}), \hat{x}}^{5d} \times |1\rangle = \langle 1| \times |Z^{5d}\rangle \\
& \langle 1| \times \prod_{\hat{x} \in \mathcal{X}_0}^{\succ} S_{i(\hat{x}), \hat{x}}^{6d} \times |1\rangle = \text{Tr} \left[p^{L_0} \times \right] \quad (8.15)
\end{aligned}$$

FIGURE 4. Conformal blocks as the partition function of five-dimensional (top) and six-dimensional theory (bottom). The six-dimensional block has two equivalent expressions.

and the commutation relation between the Υ and S operators is given by

$$[\Upsilon_i(x), S_j(x')] = \delta_{i,j} \frac{\theta(q_1^{-1}; p)}{(p; p)_\infty^2} \delta\left(q_1 \frac{x'}{x}\right) : \Upsilon_i(x) S_j(x') : \quad (8.13)$$

which becomes zero in the limit $q_1 \rightarrow 1$.

We can obtain the non-extended partition function from the Z -state as a correlator of the elliptic algebra as well as (5.16). In this case, the elliptic correlator can be also written in terms of 5d operators¹⁴

$$Z(t=0) = \langle 1| \prod_{\hat{x} \in \mathcal{X}_0}^{\succ} S_{i(\hat{x}), \hat{x}}^{6d} |1\rangle = \text{Tr} \left[p^{L_0} \prod_{\hat{x} \in \mathcal{X}_0}^{\succ} S_{i(\hat{x}), \hat{x}}^{5d} \right] \quad (8.15)$$

¹⁴To assure the modularity of (non-extended) gauge theory partition function, we need to assign the anomaly-free condition for 6d theory

$$\sum_{j \in \Gamma_0} c_{ij} n_j = n_i^{\text{flavor}} \quad \text{for } \forall i \in \Gamma_0. \quad (8.14)$$

This is equivalent to the conformal condition in 4d quiver gauge theory.

where the energy operator is defined

$$L_0 = \sum_{i \in \Gamma_0} \sum_{m=1}^{\infty} m t_{i,m} \frac{\partial}{t_{i,m}}. \quad (8.16)$$

The meaning of this trace formula is as follows. One possibility to impose the periodicity (8.1) is doubling the free fields. Another possible way is to change the definition of the correlator to the trace, as shown in the bottom of Fig. 4, as well as the usual CFT correlator on a torus. They are actually two equivalent methods to compute the torus correlation function [28, 54]. From this point of view, the Z -state itself is not necessarily modular invariant, while the torus correlator should be.

Let us show the generating current of elliptic W-algebra with A_r linear quiver examples. The generating current itself is constructed by the Y -operator in the same way as before. The elliptic $W(A_1)$ generating current is given by

$$T_1(x) = Y_1(x) + \frac{1}{Y_1(q^{-1}x)}, \quad (8.17)$$

as discussed in Sec. 6.1, and $W(A_2)$ currents are given by

$$T_1(x) = Y_1(x) + : \frac{Y_2(\mu^{-1}x)}{Y_1(q^{-1}x)} : + \frac{1}{Y_2(\mu^{-1}q^{-1}x)}, \quad (8.18)$$

$$T_2(x) = Y_2(x) + : \frac{Y_1(\mu q^{-1}x)}{Y_2(q^{-1}x)} : + \frac{1}{Y_1(\mu q^{-2}x)}, \quad (8.19)$$

where $\mu = \mu_{1 \rightarrow 2} = \mu_{2 \rightarrow 1}^{-1}q$, as shown in Sec. 6.2. The construction is also parallel for the linear quiver A_r shown in Sec. 6.3.

These generating currents exhibit a similar OPE relation. For A_1 quiver we have [51]

$$\begin{aligned} f_{11} \left(\frac{y}{x} \right) T_1(x) T_1(y) - f_{11} \left(\frac{x}{y} \right) T_1(y) T_1(x) \\ = - \frac{\theta(q_1; p) \theta(q_2; p)}{(p; p)_{\infty}^2 \theta(q; p)} \left(\delta \left(\frac{qy}{x} \right) - \delta \left(\frac{qx}{y} \right) \right) \end{aligned} \quad (8.20)$$

and for A_2 quiver we have

$$\begin{aligned} f_{11} \left(\frac{y}{x} \right) T_1(x) T_1(y) - f_{11} \left(\frac{x}{y} \right) T_1(y) T_1(x) \\ = - \frac{\theta(q_1; p) \theta(q_2; p)}{(p; p)_\infty^2 \theta(q; p)} \left(\delta \left(q \frac{y}{x} \right) T_2(\mu^{-1} x) - \delta \left(q^{-1} \frac{y}{x} \right) T_2(\mu^{-1} q x) \right) \end{aligned} \quad (8.21)$$

$$\begin{aligned} f_{12} \left(\frac{y}{x} \right) T_1(x) T_2(y) - f_{21} \left(\frac{x}{y} \right) T_2(y) T_1(x) \\ = - \frac{\theta(q_1; p) \theta(q_2; p)}{(p; p)_\infty^2 \theta(q; p)} \left(\delta \left(\mu q \frac{y}{x} \right) - \delta \left(\mu q^{-2} \frac{y}{x} \right) \right) \end{aligned} \quad (8.22)$$

$$\begin{aligned} f_{22} \left(\frac{y}{x} \right) T_2(x) T_2(y) - f_{22} \left(\frac{x}{y} \right) T_2(y) T_2(x) \\ = - \frac{\theta(q_1; p) \theta(q_2; p)}{(p; p)_\infty^2 \theta(q; p)} \left(\delta \left(q \frac{y}{x} \right) T_1(\mu q^{-1} x) - \delta \left(q^{-1} \frac{y}{x} \right) T_1(\mu x) \right) \end{aligned} \quad (8.23)$$

where

$$f_{ij}(x) = \exp \left(\sum_{n \neq 0} \frac{(1 - q_1^n)(1 - q_2^n)}{n(1 - p^n)} \tilde{c}_{ji}^{[-n]} x^n \right). \quad (8.24)$$

These OPE relations characterize the algebraic relation for the elliptic W-algebra generators.

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